Vatutin V.A., Dyakonova E.E. (Steklov Mathematical Institute of the Russian Academy of Sciences, Moscow, Russia) **Population size of critical Galton-Watson processes under small deviations and infinite variance**.

Let $Z(n), n \geq 0$, be a Galton-Watson branching process with generating function $f(s) := \mathbf{E}s^{\xi}$ of the offspring number ξ of one particle. We assume that

$$\mathbf{E}\xi = 1,\ f(s) := \mathbf{E}s^{\xi} = s + (1-s)^{1+\alpha} L(1-s), \quad 0 \leq s \leq 1, \tag{1}$$

where $\alpha \in (0,1)$ and $L(z)$ is a slowly varying function as $z \downarrow 0$. Let $f_0(s) := s$ and $f_n(s) = f(f_{n-1}(s)),\ n = 1, 2, ...$ It is known that, given (1)

$$\lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda(1-f_n(0))Z(n)} | Z(n) > 0, Z(0) = 1\right] = \int_0^{\infty} e^{-\lambda x} dM(x), \ \lambda > 0,$$

where $M(x)$ is a proper nondegenerate distribution. Let $\varphi(n),\ n = 1, 2, ...$, be a deterministic function such that

$$\varphi(n) \to \infty \text{ and } \varphi(n) = o(n) \text{ as } n \to \infty \tag{2}$$

and $\mathcal{H}(n, \varphi(n)) := \left\{0 < (1 - f_{\varphi(n)}(0))Z(n) \leq 1\right\}$.

Theorem 1 *Let conditions (1) and (2) be valid. Then, for any* $\lambda > 0$

1) if $m \to \infty$ *and* $m = \theta n + o(n)$ *for some* $\theta \in [0, 1)$ *then*

$$\lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda(1-f_m(0))Z(m)} | \mathcal{H}(n, \varphi(n))\right] = \frac{1}{\left(1 - \theta + \left(\lambda (1-\theta)^{1/\alpha} + \theta^{1/\alpha}\right)^{\alpha}\right)^{1/\alpha+1}};$$

2) if $m = n - y\varphi(n)$, *where* $y \in (0, \infty)$, *then*

$$\lim_{n\to\infty} \mathbf{E}\left[e^{-\lambda(1-f_{n-m}(0))Z(m)} | \mathcal{H}(n, \varphi(n))\right] = \sum_{j=1}^{\infty} \frac{\alpha\Gamma(j+\alpha)}{j!\,(1+\lambda)^{\alpha+j}} y M^{*j}(y^{-1/\alpha}),$$

where $M^{*j}(x)$ *is the* j*-th convolution of* $M(x)$ *with itself.*

Acknowledgement This work was supported by the Russian Science Foundation under grant no.24-11-00037 https://rscf.ru/en/project/24-11-00037/