First Passage Time Probabilities in Diffusion Models

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This work is devoted to the first passage time problem for diffusion processes. Consider a random process X_t defined by a time-homogeneous stochastic differential equation (SDE) of the form

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \quad X_0 = x,$$

where $\mu : \mathbb{R}_+ \to \mathbb{R}$ is the drift coefficient, $\sigma : \mathbb{R}_+ \to \mathbb{R}$ is the diffusion coefficient, and W_t is standard Brownian motion. We assume that the SDE admits a unique strong solution with transition probability $P_x(X_t \in A) = P(X_t \in A \mid X_0 = x)$, defined for $t \geq 0$ and any Borel set $A \subset \mathbb{R}$. Suppose that there exists a smooth transition density $p_t(y, x)$ such that $P(X_t \in A \mid X_0 = x) = \int_A p_t(y, x)\,dy$. Define the first hitting time of the continuously differentiable lower barrier $b(t)$ by

$$\tau_b = \inf\{t \geq 0 : X_t \leq b(t)\}, \quad X_0 = x > b(0).$$

Using the local time-space formula [1], we derive an integral equation for the cumulative distribution function of the hitting time $F(t, x) = P_x(\tau_b \leq t)$.

Theorem 1. *The cumulative distribution function $F(t, x)$ is given by*

$$F(t, x) = P_x(X_t \leq b(0)) + \int_0^t f(s)\, p_{t-s}(b(s), x)\, ds,$$

where the function $f(s)$ is the solution to the linear Volterra integral equation of the first kind

$$P_{b(t)}(X_t > b(0)) = \int_0^t f(s)\, p_{t-s}(b(s), b(t))\, ds.$$

A similar result can be obtained for the first hitting time of the upper boundary $b(t)$ when $X_0 < b(0)$, as well as an extension to the case of double barriers, which leads to a system of coupled Volterra integral equations.

References

[1] G. Peskir, A change-of-variable formula with local time on curves, *Journal of Theoretical Probability*, **18**:3, (2005), 499–535.