On the fractional differential Riccati equation and some new numerical approaches to its solution.

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The following fractional differential equation

$$D^{\alpha}\psi(t) = \lambda\psi^2(t) + \mu\psi(t) + \nu, \quad I_{1-\alpha}\psi = u \in \mathbb{R}, \quad \lambda, \mu, \nu \in \mathbb{R}, \quad \alpha \in (0,1], \tag{1}$$

where $D^{\alpha}\psi(t)$ represents the Riemann-Liouville fractional derivative of ψ of order α in t, is known as fractional differential Riccati equation. It appears in many different problems, as noted in [4]. For example, in the rough Heston model

$$\begin{cases} dS_t = S_t\sqrt{V_t}dW_t, \\ V_t = V_0 + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\alpha-1}\eta(m - V_s)ds + \int_0^t (t-s)^{\alpha-1}\eta\zeta\sqrt{V_s}dB_s\right). \end{cases} \tag{2}$$

which describes the dynamics of an asset price S_t and its variance process V_t. It has been shown in [5], that the characteristic function of the log-price S_t is expressed in terms of the solution of a fractional Riccati equation (2).

The fractional Riccati equation has a non-trivial solution. Some numerical approaches in solving the fractional Riccati have been elaborated. For example, through the Adomian's decomposition and the homotopy perturbation method (see [6] and references there inside). We will discuss a new approach from [1] based on the fractional power series expansion of the solution. Moreover, In recent times, Neural Networks have gained popularity, since they can be used as universal approximators of continuous functions in an interval $I \subset \mathbb{R}$ (see Universal Approximation Theorem for Neural Networks [3]). They have been used with great success in solving differential equations (ref. [2]). We will use them in the approximation of the solution to the fractional Riccati.

The general and flexible nature of Neural Networks suggests that can find applications in other problems, for example solving other fractional differential equations, which in recent times find various applications in modeling natural phenomena (ref. [4]).

References

[1] Callegaro G., Grasselli M., Pagès G. *Fast Hybrid Schemes for Fractional Riccati Equations (Rough is not so Tough)*, Mathematics of Operations Research, Vol. 46, 221-254, 2021

[2] Lagaris I. E., Likas A., Fotiadis D. I., *Artificial Neural Networks for Solving Ordinary and Partial Differential Equations* IEEE Transactions On Neural Networks, vol. 9, nr. 5, p. 989-1000, September 1998

[3] Hassoun M., *Fundamentals of Artificial Neural Networks*, MIT Press 1995

[4] Tverdyi D., Parovik R. *Application of the Fractional Riccati Equation for Mathematical Modeling of Dynamic Processes with Saturation and Memory Effect.* Fractal and Fractional 2022, 6, 163

[5] El Euch O., Rosenbaum M. *The characteristic function of rough Heston models.*. Mathematical Finance, 29(1):3–38, (2019).

[6] Rahimkhani P., Ordokhani Y., Babolian E. *Application of fractional-order Bernoulli functions for solving fractional Riccati differential equation.* Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 277-292.