Random regular flows of classical strings

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The Model Consider the set $X_{\infty} = X_{\infty}(N.L)$ of infinite periodic sequences of point particle coordinates on the real axis R

$$... < x_{-1} < x_0 < x_1 < ... < x_{N-1} < x_N < ..., \quad (1)$$

where periodic means that $x_{k+N} = x_k + L$ for any k, some fixed real $L > 0$ and integer $N > 0$. Masses are assumed to be 1.

Each particle collides with some external particle of small mass at random moments of time. Namely, we suppose, that at the random moments $\tau_{k,l}$, where $\tau_{k,1} < \tau_{k,2} < ...\tau_{k,l}...$, $k-$th particle collides with a particle of mass $m < 1$ and velocity u. Let $\sigma_{k,l} = \tau_{k,l+1} - \tau_{k,l} > 0$ be independent exponentially distributed random variables. After a collision the velocity of the particle changes, according to the laws of conservation of momentum and energy, as follows:

$$v(\tau_{k,l}) = bv(\tau_{k,l}-) + (1-b)u, \ b = \frac{1-m}{1+m}$$

We assume that between collisions the dynamics is deterministic and is defined by Newton's equations

$$\ddot{x}_k = \omega^2(x_{k+1} - 2x_k + x_{k-1}) + f_k, \quad (2)$$

with external (driving) forces $f_k \equiv f = const$, and formal interaction potential energy

$$U = \frac{\omega^2}{2} \sum (x_{k+1} - x_k - \frac{L}{N})^2, \quad (3)$$

In periodic case initial conditions (and the dynamics itselth) are in fact finite dimensional - one can assume that at time 0 there are exactly N point particles $0, 1, ..., N-1$ with velocities $v_0(0), ..., v_{N-1}(0)$ coordinates inside $[0, L)$:

$$0 = x_0(0) < x_1(0) < ... < x_{N-1}(0) < x_N(0) = L, \quad (4)$$

Let $x_k^{(N)}(t)$ be the position of $k-$th particle at time t. The defined random process $(x_0^{(N)}(t), x_1^{(N)}(t), ..., x_{N-1}^{(N)}(t))$ is a piecewise-deterministic continuous time Markov process with trajectories continuous from the right.

Let $\omega = \omega_0 N$. We shall assume, that $\varepsilon = E\sigma_{1,1} \to 0$, such that $\frac{2m}{\varepsilon} \to \alpha > 0$ for some constant α and $\varepsilon N \to 0$.

Regularity conditions We call the dynamics **regular** (without collisions) if it cannot occur that $x_{k+1}(t) = x_k(t)$ for some k and $t \geq 0$. In regular dynamics the order of particles is conserved. We shall say that periodic initial conditions have "almost smooth profiles" if the following two conditions hold:

1) there exist smooth enough periodic functions $X(x), V(x))$ with period L, where $X(x) > 0$ for any $x \in R$, and

$$\int_0^L X(u)du = L, \ \int_0^L V(u)du = 0 \quad (5)$$

2) for some constants $C_1 > 0, C_2 > 0$

$$|x_{k+1}^{(N)}(0) - x_k^{(N)}(0) - \frac{L}{N}X(\frac{kL}{N})| < \frac{C_1}{N^2}, |\dot{x}_{k+1}^{(N)}(0) - \dot{x}_k^{(N)}(0) - \frac{L}{N}V(\frac{kL}{N})| < \frac{C_2}{N^2} \quad (6)$$

uniformly in k.

Define constants

$$c_1 = L\int_0^L |\frac{d^2X}{du^2}(u)|du, c_2 = L\int_0^L |\frac{d^2V}{du^2}(u)|du, \quad (7)$$

In some sense c_1, c_2 define fluctuations of the "profile". We will need also the constant

$$\gamma = \gamma(X, V, \alpha, \omega_0, C_1, C_2, c_1, c_2) = (1 + \frac{\alpha}{8\omega_0})(2c_1 + C_1L^{-1}) + \frac{2c_2 + C_2L^{-1}}{4\omega_0} > 0 \quad (8)$$

Let $\Omega_N^{(0)}(\delta) \subset R^{2N}, 0 < \delta < 1$, be a set of "almost smooth" initial conditions $x_k^{(N)}(0), \dot{x}_k^{(N)}(0), k = 0, ..., N-1$, with additional condition that $\gamma(X, V) < \delta$, and $\Omega_N(\delta)$ be the domain of $R^N = \{(x_0, ..., x_{N-1})\}$, defined for some $0 < \delta < 1$ by the estimates

$$|x_{k+1} - x_k - \frac{L}{N}| < \frac{L\delta}{N}$$

for all k.

Theorem 1 *Let initially the system belong to $\Omega_N^{(0)}(\delta)$ for some $0 < \delta < 1$. Then with probability 1 it stays in $\Omega_N(\delta)$ for all $t \geq 0$, that is*

$$|x_{k+1}^{(N)}(t) - x_k^{(N)}(t) - \frac{L}{N}| < \frac{L\delta}{N}$$

for all k, t.

It follows that with probability 1 particles conserve the initial order at any time $t > 0$.

Convergence to regular continuum mechanics Let $x_k^{(N)}(t)$ be the position of $k-$th particle at time t.

With each point $x \in R$ we associate the particle with number $k(x, N)$ such that

$$x_{k(x,N)}^{(N)}(0) \leq x < x_{k(x,N)+1}^{(N)}(0)$$

Theorem 2 *We have*

1)For any $T > 0$ uniformly in $t \in [0, T]$ and in $x \in R$ there exists the limit almost surely

$$\lim_{N \to \infty, \varepsilon \to 0} x_{k(x,N)}^{(N)}(t) = Y(t, x) \in R \tag{9}$$

where function $Y(t, x)$ satisfies the condition $Y(t, x + L) = Y(t, x) + L$ for any $x \in R$.

2) Moreover, $Y(t, x) : R \to R$ is differentiable in x and t and strictly increasing in x for each fixed t. So it is a diffeomorphism of R for any t .

The function $Y(t, x) \in R$ will be called the trajectory of the continuous media particle which is initially at point $x \in R$.

Conservation law, Euler equation and pressure For given N define the distribution function on $[0, L)$

$$F^{(N)}(t, y) = \frac{1}{N} \sharp\{k \in \{0, 1, ..., N-1\} : \pi(x_k(t)) \leq y\}, y \in [0, L)$$

where $\pi(x) = x, \mod L$. One can prove that uniformly in $y \in [0, L)$ and in $t \in [0, T]$, for any $T < \infty$, we have almost surely

$$\lim_{N \to \infty, \varepsilon \to 0} F^{(N)}(t, y) = F(t, y), y \in [0, L),$$

where $\varepsilon \to 0, m \to 0, N \to \infty$, such that $\frac{2m}{\varepsilon} \to \alpha$, $\varepsilon N \to 0$ and $F(t, y)$ is twice differentiable in y and t.

Define the density of "the number of continuum media particles" as

$$\rho(t, y) = \frac{dF(t, y)}{dy}, y \in [0, L) \tag{10}$$

As the particles do not collide, then one can unambiguously define the function $u(t, y)$ as the speed of the (unique) particle situated at time t at the point y.

Theorem 3 *Denote $\omega_1 = \omega_0 L$. For any $t > 0$, $y \in [0, L)$ we have :*

$$\frac{\partial \rho(t, y)}{\partial t} + \frac{d}{dy}(u(t, y)\rho(t, y)) = 0 \tag{11}$$

$$\frac{\partial u(t, y)}{\partial t} + u(t, y)\frac{\partial u(t, y)}{\partial y} + \alpha u(t, y) - f = -\frac{\omega_1^2 \rho_y(t, y)}{\rho^3(t, y)} = \frac{1}{\rho(t, y)} \frac{d}{dy} \frac{\omega_1^2}{\rho(t, y)} = -\frac{p_y(t, y)}{\rho(t, y)} \tag{12}$$

where $p(t, y)$ is called pressure and is defined as follows:

$$p(t, y) = -\frac{\omega_1^2}{\rho(t, y)} + C \tag{13}$$

and C is a constant.