SOME RESULTS ON SIGNED INTERPOLATING DEFLATORS

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Currently, the development of the theory of Haar interpolations of financial markets with the use of martingale measures continues. The existence of martingale measures of discounted stock prices means that this kind of interpolation can only be used in complete markets. However, real financial markets often contain elements of arbitrage opportunities. Therefore, it is important to develop techniques for interpolating processes that do not admit martingale measures. This work is just devoted to this problem. Here, signed deflators serve as the main interpolation tool. With their help, the Haar interpolation procedure is defined. In the case of the existence of martingale measures, this procedure leads to the process interpolation, which coincides with the martingale interpolation. The paper introduces the concept of an admissible deflator, defines (as when martingale measures exist) the universal Haar uniqueness property and its weakened variants. The main results of the work are related to the so-called special Haar uniqueness property, which leads to the uniqueness of the admissible deflator.

Consider a stochastic basis $\big(\Omega, F=(\mathcal{F}_k)_{k=0}^K, P\big)$, where Ω be a set, $F=(\mathcal{F}_k)_{k=0}^K$ be a strictly increasing filtration, $\mathcal{F}_0=\{\Omega,\emptyset\}$, $K\le\infty$, any $\mathcal{F}_k$ $(0\le k<K+1)$ be finite, and P be a probability on $\mathcal{F}_K$ (if $K=\infty$, then $\mathcal{F}_K=\mathcal{F}_\infty$ is the least σ-algebra containing all $\mathcal{F}_k$, $0\le k<\infty$). We assume that the probability measure P loads all non-empty subsets from $\mathcal{F}_k$, $0\le k<K+1$.

Let $Z=(Z_k,\mathcal{F}_k)_{k=0}^K$ be an adabted process that can take any real values. A martingale $D=(D_k,\mathcal{F}_k,P)_{k=0}^K$ is said a signed deflator of the process Z if D_0=1 and the process $DZ=(D_kZ_k,\mathcal{F}_k,P)_{k=0}^K$ is a martingale.

We use in the sequel the following system of notations. Let A be an atom in $\mathcal{F}_k$, B_i $(i=1,2,\dots,m)$ be atoms in $\mathcal{F}_{k+1}$,

$$A=B_1+B_2+\cdots+B_m, a:=Z_k|_A, b_i:=Z_{k+1}|_{B_i}, p_i:=P(B_i), d_i:=D_{k+1}|_{B_i}.$$

Generally splitting index m of atom A and numbers a, b_i, p_i, d_i depend on A.

A signed deflator D of the process Z is said admissible if $\forall 0\le k<K+1$, for all atom $A\in\mathcal{F}_k$ and for all non-empty subset $I\subset\{1,2,\dots,m\}$

$$\sum_{i\in I}p_id_i\neq 0.$$

We will also consider on $(\Omega,\mathcal{F}_K)$ Haar filtrations (HF)

$$H=(\mathcal{H}_n)_{n=0}^L,\ \mathcal{H}_n\subset\mathcal{F}_K, \tag{0.1}$$

where $\mathcal{H}_0=\{\Omega,\emptyset\}$ and each σ-algebra $\mathcal{H}_n$ is generated by a partition of the set Ω into exactly $n+1$ atoms $H_0^n, H_1^n, ..., H_n^n$. A Haar filtration is said special

Haar filtration if at every moment $n > 1$ only those two atoms of $H_0^n, H_1^n, ..., H_n^n$ can be divided that were obtained by division at the previous moment $n-1$. Haar filtration $(\mathcal{H}_n)_{n=0}^L$ from (0.1) is said interpolating Haar filtration (IHF) of $(\mathcal{F}_k)_{k=0}^K$ if there exists an increasing sequence of integers $n_k, 0 \le k < K+1$, such that $\mathcal{H}_{n_k} = \mathcal{F}_k$ (and hence $\mathcal{H}_L = \mathcal{F}_K$). Special interpolating Haar filtration (SIHF) is defined analogically.

Let us fix an IHF $(\mathcal{H}_n)_{n=0}^L$ of $(\mathcal{F}_k)_{k=0}^K$ and let $D = (D_k, \mathcal{F}_k, P)_{k=0}^K$ be a signed admissible deflator of the process $Z = (Z_k, \mathcal{F}_k)_{k=0}^K$. Denoting $X_{n_k} := D_k Z_k$ and $Y_{n_k} := D_k$, we obtain martingales $(X_{n_k}, \mathcal{H}_{n_k}, P)_{k=0}^K$ and $(Y_{n_k}, \mathcal{H}_{n_k}, P)_{k=0}^K$. Then we can define two martingales $X = (X_n, \mathcal{H}_n, P)_{n=0}^L$ and $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$ in the following obvious way: for any $n < L+1$ find $n_k \ge n$ and put

$$X_n := E^P[X_{n_k}|\mathcal{H}_n],\ Y_n := E^P[Y_{n_k}|\mathcal{H}_n]. \tag{0.2}$$

It is clear that such definitions are correct.

The process $Z^{int} = (Z_n^{int}, \mathcal{H}_n)_{n=0}^L$ defined by the formula

$$Z_n^{int} = \begin{cases} Z_k,\ if\ n = n_k\ (0 \le k < K+1), \\ \dfrac{X_n}{Y_n},\ if\ n \ne n_k, Y_n \ne 0. \end{cases} \tag{0.3}$$

will be called H-interpolation of the process Z with the help of the deflator D.

It is clear that the process $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$ is a signed admissible deflator of the process $Z^{int} = (Z_n^{int}, \mathcal{H}_n)_{n=0}^L$.

Let the process $Z = (Z_k, (\mathcal{F}_k)_{k=0}^K)$ admit a martingale measure Q, equivalent to the physical measure P, i.e. the process $(Z_k, \mathcal{F}_k, Q)_{k=0}^K$ be a martingale. Denote $h := \dfrac{dQ}{dP}$ and $D_k := E^P[h|\mathcal{F}_k]$. It is clear that the process $D = (D_k, \mathcal{F}_k)_{k=0}^K$ is a strictly positive deflator of the process Z. Hence for all $n \le n_k$ $Y_n = E^P[Y_{n_k}|\mathcal{H}_n] = E^P[D_k|\mathcal{H}_n] > 0$ and $Z_n^{int} = \dfrac{X_n}{Y_n}$. Applying the generalized Bayes formula, it is easy to see that the process $(Z_n^{int}, \mathcal{H}_n, Q)_{n=0}^L$ is a martingale. From this fact it follows that H-interpolation of the process Z with the help of deflator D coincides with the Haar interpolation of Z with respect to the martingale measure Q (c.f. [1], [2]).

We say that a signed admissible deflator $D = (D_k, \mathcal{F}_k, P)_{k=0}^K$ satisfies the Haar uniqueness property (HUP) if there exists a Haar interpolation $H = (\mathcal{H}_n)_{n=0}^L$ of the initial filtration F such that the process (0.3) admits only one deflator, namely the deflator $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$, defined by (0.2).

We say that a signed deflator $D = (D_k, \mathcal{F}_k, P)_{k=0}^K$ satisfies the universal Haar uniqueness property — UHUP (resp., the special Haar uniqueness property — SHUP) if for every interpolating (resp., special interpolating) Haar filtration $H = (\mathcal{H}_n)_{n=0}^L$ of the initial filtration F the process (0.3) admits only one deflator, namely the deflator $Y = (Y_n, \mathcal{H}_n, P)_{n=0}^L$, defined by (0.2).

Theorem 1. *Let* $\forall k : 0 \le k < K+1$ *and for all atom* $A \in \mathcal{F}_k$ *we have* $m \ge 3$. *If there exists an admissible signed deflator* D *satisfying SHUP, then the numbers* $a, b_1, \ldots, b_m$ *are different.*

Theorem 2. *Let $\forall k : 0 \leq k < K+1$ and for all atom $A \in \mathcal{F}_k$ we have $m \geq 4$ and the numbers $a, b_1, \ldots, b_m$ be different. Then there exists an admissible signed deflater D satisfying SHUP.*

The problem of the existence of admissible deflators satisfying UHUP will be considered too.

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