Development of a model with random priorities

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The main result of this paper is the proof of the strict concavity of some function of integral form depending on n random variables, which we call priorities. This function is an objective function in the so-called model with priorities, in which the arbiter, following expert opinions, distributes funds among the enterprises and institutions under his jurisdiction. This result implies an important corollary about the existence and uniqueness of a local maximum point (which is also a global maximum point) of the objective function. This is a significant generalization of the corresponding result [1]. Earlier, the theorem on the existence and the uniqueness of a local maximum point (which is simultaneously a global maximum point) of the objective function was proved in paper [2], where the independence of n priorities under consideration was assumed. In this paper, on a general probabilistic model with the use of convexity arguments, an existence and uniqueness theorem is obtained under weaker (in comparison with [1]) conditions on three considered priorities.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider the function

$$\Phi(u_1, u_2, \dots, u_n) = E\left(u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n}\right), u_1 \geq 0, u_2 \geq 0, \dots, u_n \geq 0,$$

where E is the expectation with respect to the probability P; $\alpha_1, \alpha_2, \dots, \alpha_n$ are random variables taking values with probability one on the segment $[0,1]$; $u_n = -c_1u_1 - c_2u_2 - \cdots - -c_{n-1}u_{n-1} + c_n$, where $c_1, c_2, \dots, c_n$ are strictly positive parameters. We assume equal to 0 the value of $u_i^{\alpha_i}$ if $u_i = 0$, $i = 1,2, \dots, n$. Thus, we get a function of $n-1$ variables

$$F(u_1, u_2, \dots, u_{n-1}) := \Phi(u_1, u_2, \dots, u_{n-1}, -c_1u_1 - c_2u_2 - \cdots - c_{n-1}u_{n-1} + c_n)$$

defined on the domain

$$D: \begin{cases} u_1 \geq 0 \\ u_2 \geq 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \\ u_{n-1} \geq 0 \\ c_1u_1 + c_2u_2 + \cdots + c_{n-1}u_{n-1} \leq c_n, \end{cases}$$

The interior of D is denoted by D^0.

Function F is continuous on D, infinitely differentiable and strictly positive on D^0 and equal to 0 on $D \setminus D^0$. In this work we find the conditions on the priorities $\alpha_1, \alpha_2, \dots, \alpha_n$, under which the function F is strictly concave, and study some consequences of this fact.

Theorem 1. If P-almost surely (a.s.)

$$\alpha_i \geq 0 \ (i = 1,2, \dots, n), \tag{1}$$

$$\sum_{\substack{1\le k\le n,\\ k\neq i}} a_k \le 1,\ i = 1,2,\dots,n, \qquad (2)$$

then the function F is concave on D^0. If the inequalities (1) and (2) are strict, then the function F is strictly concave on D^0.

It follows from Theorem 1 the impotent result on the uniqueness of maximal point of the objective function F.

Theorem 2. If P-a.s. the conditions (1) and (2) in the strict forms are satisfied, then the function F has exactly one local (and simultaneously global) maximum point on D^0.

Remark. Let $\Omega = [0,1]$, $\mathcal{F}$ be the σ-field of Borel subsets on $[0,1]$, $dP = dx$ be the Lebesgue measure on $(\Omega, \mathcal{F})$. In [1, Proposition 5], within the framework of this model for $n = 3$, the following result was obtained: if a.e. on $[0,1]$ $0 < \alpha_1 \le \frac{1}{2}, 0 < \alpha_2 \le \frac{1}{2}, 0 < \alpha_3 \le \frac{1}{2}$, any stationary point $(u_1, u_2) \in D^0$ of function $F(u_1, u_2)$ is a point of local maximum. It can be shown that this point is unique and is also a global maximum point. In our general case we obtain all this automatically (these conditions entail the fulfillment of the conditions (1) and (2)). But the conditions of the Theorem 2 themselves are much less restrictive.

References

[1] Neumerzhitskaia N V, Uglich S I, Volosatova T A 2020 Sufficient conditions for the uniqueness of the maxima of the optimization problem in the framework of a stochastic model with priorities depending on one random variable *E3S Web of Conferences 224,* 01014 https://doi.org/10.1051/e3sconf/202022401014

[2] Pavlov I V and Uglich S I 2017 Optimization of complex systems of quasilinear type with several independent priorities *Vestnik RGUPS* **3** pp 140–145.