LIMITING THEOREM FOR SPECTRA OF ADJACENCY AND LAPLASE MATRICES OF RANDOM GRAPHS

A. TIKHOMIROV[2]

We consider not oriented simple graph (without loops and with simple edges) V, E with vertices $|V| = n$ and set of edges E such that edges $e \in E$ are independent and have probability p_e and weight w_e. Consider the adjacency $n \times n$ matrix

$$\mathbf{A} = \left[A_{jk}\right], \text{ where } A_{jk} = \begin{cases} 0, & \text{if } (j,k) \notin E, \\ 1, & \text{if } (j,k) \in E. \end{cases}$$

Define degree of vertex $i \in V$ as $d_i := \sum_{j:\{i,j\}\in E} \xi_{ij}$. We shall assume that A_{ij} for $1 \le i \le j \le n$ are independent and $\mathbb{E}A_{ij} = p_{ij}(n) =: p_{ij}$.

We introduce the diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_n)$ and Laplace matrix of not weited graph G, $\mathbf{L} = \mathbf{D} - \mathbf{A}$. We shall assume that matrix $\mathbf{A}$ is symmetry, i.e. $A_{ij} = A_{ji}$, and that r.v.'s A_{ij} for $1 \le i \le j \le n$ are independent. We shall consider as well weigthed graphs $\widetilde{G} = (V, E, w)$ with weight function $w_{ij} = w_{ji} = X_{ij}$ for $1 \le i \le j \le n$ independent random variables s.t. $\mathbb{E}X_{ij} = 0, \quad \mathbb{E}X_{ij}^2 = \sigma_{ij}$. We introduce the quanities

$$a_n = \frac{1}{n}\sum_{i,j=1}^n p_{ij}\sigma_{ij}^2, \text{ and } \widehat{a}_n = \frac{1}{n}\sum_{i,j=1}^n p_{ij}(1-p_{ij}).$$

The quantity a_n intereprete as expected mean degree of weighted graph $\widetilde{G}$. With graph $\widetilde{G}$ we consider the adjancy matrix $\widetilde{\mathbf{A}} = \left[A_{ij}X_{ij}\right]$ and Laplase or Markov matrix $\widetilde{\mathbf{L}} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{A}}$, where $\widetilde{\mathbf{D}} = \operatorname{diag}(\widetilde{d}_1, \dots, \widetilde{d}_n)$ and $\widetilde{d}_i = \sum_{j:j\ne i} A_{ij}X_{ij}$. We shall denote by $\lambda_1(\mathbf{B}) \ge \lambda_2(\mathbf{B}) \ge \dots \ge \lambda_n(\mathbf{B})$ ordered eigenvalues of symmetric $n \times n$ matrix $\mathbf{B}$. We shall consider spectrum of matrix $\frac{1}{\sqrt{a_n}}\widetilde{\mathbf{A}}$, $\frac{1}{\sqrt{a_n}}\widetilde{\mathbf{L}}$, $\widehat{\mathbf{A}} = \frac{1}{\sqrt{\widehat{a}_n}}(\mathbf{A} - \mathbb{E}\mathbf{A})$ and $\widehat{\mathbf{L}} = \frac{1}{\sqrt{\widehat{a}_n}}(\mathbf{L} - \mathbb{E}\mathbf{L})$. For bravity of notation we shall write $\widetilde{\lambda}_j = \lambda_j(\widetilde{\mathbf{A}})$, $\widehat{\lambda}_j = \lambda_j(\widehat{\mathbf{A}})$, $\widetilde{\mu}_j = \lambda_j(\widetilde{\mathbf{D}})$, and $\widehat{\mu}_j = \lambda_j(\widehat{\mathbf{D}})$. Introduce corresponding empirical spectral distributions

$$\widehat{F}_n(x) := \frac{1}{n}\sum_{j=1}^n \mathbb{I}\{\widehat{\lambda}_j \le x\}, \quad \widetilde{F}_n(x) := \frac{1}{n}\sum_{j=1}^n \mathbb{I}\{\widetilde{\lambda}_j \le x\},$$

$$\widehat{G}_n(x) := \frac{1}{n}\sum_{j=1}^n \mathbb{I}\{\widehat{\mu}_j \le x\}, \quad \widetilde{G}_n(x) := \frac{1}{n}\sum_{j=1}^n \mathbb{I}\{\widetilde{\mu}_j \le x\}.$$

In the paper [1], 2006, was shown that under condition $p_{ij} \equiv 1$ and $\sigma_{ij}^2 \equiv 1$ for any $1 \le i, j \le n$ that ESD $G_n(x)$ weakly convergence in probability to the nonrandom

Key words and phrases. Wigner law, random graph, normal law, Stieltjes transform
.

2)Institute of Physics and of Mathematics, Komi Science Center of Ural Division of RAS, Syktyvkar State University, Russia

distribution function $G(x)$, which defined as free convolution of Gaussian distribution function and semicircular distribution function.

In [3], 2010, authors considered the limit of $G_n(x)$ for weighted Erdös – Renyi graphs ($p_{ij} \equiv p_n$) and equivariance weights ($\sigma_{ij} \equiv \sigma^2$). Assuming that p_n bounded away from zero and one, and that random variables X_{ij} have the fourth moment, proved that $G_n(x)$ weakly convergence to the same function $G(x)$.

In [5], 2020, Yizhe Zhu consider the so calle graphon approach (the descrition see below) to limiting spectral distribution of Wigner–type matrices. He discribe the moments of limit spectral measure in term of graphon of profile of variance matrix $\Sigma = (\sigma_{ij})$ and number of trees with fixed number of vertices. Recently Chatterjee and Hazra published the paper [2] in which developed the approach of Zhu.

First we formulate some conditions which we shall use in the present paper.

- Condition $CP(0)$: $a_n \to \infty,$ as $n \to \infty$ and
$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left|\frac{1}{a_n} \sum_{j=1}^{n} p_{ij}\sigma_{ij}^2 - 1\right| = 0, and$$
- Condition $CX(1)$: For any $\tau > 0$
$$L_n(\tau) := \frac{1}{a_n} \sum_{i,j=1}^{n} p_{ij}\mathbb{E}X_{ij}^2 \mathbb{I}\{|X_{ij}| > \tau\sqrt{a_n}\} \to 0 \text{ as } n \to \infty, \tag{0.1}$$

The main result of the present paper is the following theorem.

Theorem 0.1. *Let conditions $CP(0)$, $CX(1)$ hold. Then*

- *ESD's $F_n(x)$ weakly convergence in probability to the semi-circular distribution function,*
$$\lim \widetilde{F}_n(x) = F(x) \textit{ and } \lim \widehat{F}_n(x) = F(x) \textit{ in probability.}$$
- *ESD's $\widetilde{G}_n(x)$ convergence in probability to the distribution function $G(x)$, which is additive free convolution of standard normal distribution function and semi-circular distribution function,*

$$\lim \widetilde{G}_n(x) = G(x) \textit{ and } \lim \widetilde{G}_n(x) = G(x) \textit{ in probability..}$$

References

[1] Wloodzimer Bryc, Amir Dembo, Tiefeng Jiang (2006) Spectral Measure of Large Random Hankel, Markov and Toeplitz Matrices. *The Annals of Probability*, **34, No 1, 1–38.**

[2] Anirban Chatterjee, Rajat Subhra Hazra (2021) Spectral Properties for the Laplacian of a Generalized Wigner Matrices. **Arxiv:2011.07912v2**

[3] Xue Ding, Tiefeng Jiang (2010) Spectral Distribution of Adjancency Matrix and Laplacian Matrices of Random Graphs. *The Annals of Applied Probability*, **vol. 20, No. 6, 2086–2117**

[4] Fernando L. Metz, Jeferson D. Silva (2020) Spectral density of dense networks and the breakdown of the Wigner semicircle law. *Physical review research*,**vol.2**, 043116.

[5] Yizhe Zhu (2020) A Graphon Approach to Limiting Spectral Distribution of Wigner-Type Matrices. *Random Structure&Algorithms*, **56(1), 251–279**

[6] A.N. Tikhomirov (2021). On the Wigner law for Generalized Erdös–Renyi Random Graphs. *Siberian Advances in Mathematics,* **vol. 31, No. 4, pp. 229–236.**