Residual Autocovariances Tests for Autoregressive Models

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Consider stationary autoregressive model AR(p) of order p:

$$u_t = \beta_1 u_{t-1} + \cdots + \beta_p u_{t-p} + \varepsilon_t, \quad t \in \mathbb{Z},$$

where $\{\varepsilon_t\}$ are i.i.d. random variables with an unknown distribution function G and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)'$ are unknown parameters such that the roots of the characteristic equation $x^p = \beta_1 x^{p-1} + \cdots + \beta_p$ lie inside the unit circle.

Given the observations $u_{1-p}, \ldots, u_n$ a new class of robust tests for $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$ is introduced: tests based on the residual autocovariances,

$$\Gamma_{tn}^{\varphi\psi}(\boldsymbol{\beta}) := \sum_{k=1}^{n-t} \varphi(\varepsilon_k(\boldsymbol{\beta}))\psi(\varepsilon_{k+t}(\boldsymbol{\beta}))$$

for some special choice of functions φ, ψ, where residuals $\varepsilon_t(\boldsymbol{\beta}) := u_t - \beta_1 u_{t-1} - \cdots - \beta_p u_{t-p}$. Define the sequence $\{\delta_t(\boldsymbol{\beta})\}$ using the recurrent formula $\delta_t(\boldsymbol{\beta}) = \beta_1 \delta_{t-1} + \cdots + \beta_p \delta_{t-p}$, $t \in \mathbb{N}$, with initial conditions $\delta_{1-p} = \cdots = \delta_{-1} = 0$, $\delta_0 = 1$. Then the RA test statistic is the p-vector

$$\boldsymbol{T}_n^{\varphi\psi}(\boldsymbol{\beta}_0) := \sum_{k=1}^{n-t} \Gamma_{tn}^{\varphi\psi}(\boldsymbol{\beta}_0)(\delta_{t-1}(\boldsymbol{\beta}_0), \ldots, \delta_{t-p}(\boldsymbol{\beta}_0))'.$$

The RA estimates for ARMA models were introduced by Bustos and Yohai [2]. The robust test based on the sign autocovariances (for the case $\varphi = \psi = sign$) was investigated by Boldin, Simonova, Tyurin [1]. The RA tests for AR(1) was described in [4]. Here we present more general case for AR(p).

Theorem 1 *Let $\varphi(\varepsilon_1), \psi(\varepsilon_1)$ have zero means and finite nonzero variances. Then under $H_0 : \boldsymbol{\beta} = \boldsymbol{\beta}_0$,*

$$n^{-1/2}\boldsymbol{T}_n^{\varphi\psi}(\boldsymbol{\beta}_0) \xrightarrow{\text{d}} \boldsymbol{N}^p(\boldsymbol{0}, \mathbb{E}\,\varphi^2(\varepsilon_1)\,\mathbb{E}\,\psi^2(\varepsilon_1)\boldsymbol{K}(\boldsymbol{\beta}_0)), \quad n \to \infty,$$

where $\boldsymbol{K}(\boldsymbol{\beta}) := \sum_{t=1}^{\infty}(\delta_{t-1}(\boldsymbol{\beta}), \ldots, \delta_{t-p}(\boldsymbol{\beta}))'(\delta_{t-1}(\boldsymbol{\beta}), \ldots, \delta_{t-p}(\boldsymbol{\beta}))$.

In order to study asymptotic relative efficiency of our test relative to known tests, we need limiting distribution of the test statistic under local alternatives. The result is obtained via Le Cam's Third Lemma and sufficient condition for the LAN property is given by Kreiss [3].

Theorem 2 *Let the distribution G of ε_1 possesses an absolutely continuous Lebesgue density $g(x) > 0$. Let $\varepsilon_1, g'/g(\varepsilon_1), \varphi(\varepsilon_1), \psi(\varepsilon_1)$ have zero means and finite nonzero variances. Then under the sequence of local alternatives $H_{1n} : \boldsymbol{\beta} = \boldsymbol{\beta}_n = \boldsymbol{\beta}_0 + n^{-1/2}\boldsymbol{\tau}$,*

$$n^{-1/2}\boldsymbol{T}_n^{\varphi\psi}(\boldsymbol{\beta}_0) \xrightarrow{\text{d}} \boldsymbol{N}^p(-\,\mathbb{E}\,\varphi(\varepsilon_1)\varepsilon_1\,\mathbb{E}\,\psi(\varepsilon_1)g'/g(\varepsilon_1)\boldsymbol{K}(\boldsymbol{\beta}_0)\boldsymbol{\tau}, \mathbb{E}\,\varphi^2(\varepsilon_1)\,\mathbb{E}\,\psi^2(\varepsilon_1)\boldsymbol{K}(\boldsymbol{\beta}_0)), \quad n \to \infty.$$

We also perform a simulation study to show the robustness properties of RA tests with bounded φ, ψ.

References

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