Gushchin A.A. (Moscow, Russia) — **Joint distributions of increasing processes and their compensators, single jump martingales, and the Skorokhod embedding.**

Let us denote by $\mathbb{W}^*$ the class of all Borel probability measures $\mu = \mu(dx, dy)$ on $\mathbb{R}^2_+$ satisfying

$$\int x\,\mu(dx,dy) = \int y\,\mu(dx,dy) \quad \text{and} \quad \int_{\{y\leqslant\lambda\}} x\,\mu(dx,dy) \leqslant \int (y\wedge\lambda)\,\mu(dx,dy) \quad \forall\lambda\geqslant 0.$$

If, additionally, we have equality in the inequalities for all $\lambda \geqslant 0$, then the corresponding class is denoted by $\mathbb{W}^*_e$.

Let us recall that, for an increasing process $A = (A_t)_{t\geqslant 0}$, the family of random variables $C_t = \inf\{s \geqslant 0\colon A_s > t\}$ is the change of time generated by A. For a progressively measurable process X, the time-changed process $X \circ C = (X_{C_t})_{t\geqslant 0}$ is well defined if X_t converges a.s. to a finite limit X_∞ as $t \to \infty$ on the set $\{A_\infty < \infty\}$.

Here a single jump martingale is understood in a narrow sense, namely, $M = (M_t)_{t\geqslant 0}$, is said to be a single jump martingale if it has the form

$$M_t = W \wedge t - V\mathbf{1}_{\{t\geqslant W\}},$$

where a pair (V, W) of random variables has a joint distribution from $\mathbb{W}^*_e$. It is easy to show that then M is indeed a martingale (e.g. with respect to the filtration that it generates). Let us note that the process $A_t := W \wedge t$ is continuous and hence predictable, therefore, it is the compensator of the increasing process $X_t := V\mathbf{1}_{\{t\geqslant W\}}$.

The following proposition is the initial point for further analysis.

Proposition 1 ([1]) *Let X be a nonnegative local submartingale with the Doob–Meyer decomposition $X = M + A$, $X_0 = M_0 = A_0 = 0$. If $\mathbf{P}(A_\infty < \infty) = 1$, then X_t converges a.s. to a finite limit X_∞ as $t \to \infty$ and $\mathrm{Law}(X_\infty, A_\infty) \in \mathbb{W}^*$. Moreover, $\mathrm{Law}(X_\infty, A_\infty) \in \mathbb{W}^*_e$ if and only if $(X - A) \circ C$ is a single jump martingale, where C is the change of time generated by A.*

In particular, if N, $N_0 = 0$, is a local martingale such that its running maximum $\overline{N}_t := \sup_{s\leqslant t} N_s$ is continuous and $\mathbf{P}(\overline{N}_\infty < \infty) = 1$, then $\mathrm{Law}(\overline{N}_\infty - N_\infty, \overline{N}_\infty) \in \mathbb{W}^*$. Moreover, for any $\mu \in \mathbb{W}^*_e$, there is a single jump martingale M such that $\mathrm{Law}(\overline{M}_\infty - M_\infty, \overline{M}_\infty) = \mu$. Therefore, for any local martingale satisfying the above conditions and such that $\mathrm{Law}(\overline{N}_\infty - N_\infty, \overline{N}_\infty) \in \mathbb{W}^*_e$, there exists a single jump martingale M such that the joint distribution $\mathrm{Law}(M_\infty, \overline{M}_\infty)$ coincides with $\mathrm{Law}(N_\infty, \overline{N}_\infty)$. On the other hand, if we embed M into the Brownian motion according to the first Monroe theorem, one can construct a Skorokhod embedding τ, i.e. a Brownian motion B and a minimal stopping time τ such that the joint distribution $\mathrm{Law}(B_\tau, \overline{B}_\tau)$ is again the same as $\mathrm{Law}(N_\infty, \overline{N}_\infty)$.

There appears a natural question if there are similar representations for distributions in $\mathbb{W}^*$. It turns out that, for any $\mu \in \mathbb{W}^*$, one can construct a locally integrable increasing process X with a continuous compensator A and such that $\mathrm{Law}(X_\infty, A_\infty) = \mu$. However, X can be chosen as an increasing process with a single jump if and only if

$$\int (x-y)^+\,\mu(dx,dy) \geqslant \int (y-x)^+\,\mu(dx,dy).$$

REFERENCES

1. *Gushchin A.A.* The joint law of terminal values of a nonnegative submartingale and its compensator, Theory of Probability and its Applications, 2018, vol. 62, № 2, pp. 216–235.

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